Biconditionals (Ch 2.6)

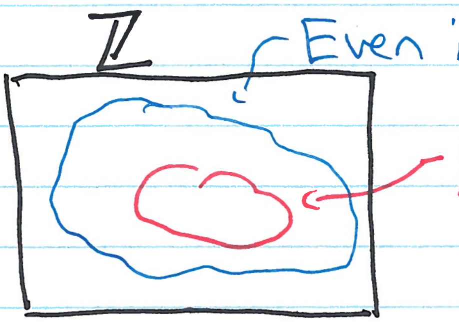
Typically we can have statements P & Q such that

$$P \Rightarrow Q \quad \text{but} \quad Q \not\Rightarrow P$$

Example: For an integer n,

$$n \text{ is a multiple of } 6 \Rightarrow n \text{ is even}$$

$$n \text{ is even} \not\Rightarrow n \text{ is a multiple of } 6$$



- "$6|n$" is a "stronger" or "more restrictive" condition
- "$2|n$" is a "weaker" or "more relaxed"

Def: Suppose $P \Rightarrow Q$ and $Q \Rightarrow P$.

Then "P is true if and only if Q is true"

"P iff Q"

$$P \Leftrightarrow Q$$

P	Q	$P \iff Q$
T	T	T ← True/True
T	F	F
F	T	F
F	F	T ← False/False.

<u>Fact:</u>

$$(P \iff Q) \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$$

$$\equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

<u>Logical derivation:</u>

$$(P \iff Q) \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

$$\equiv (\sim P \vee Q) \wedge (\sim Q \vee P)$$

$P \Rightarrow Q \equiv \sim P \vee Q$

(Distributive)

$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

$$\equiv \cancel{(\sim P \wedge P)} \vee (\sim P \wedge \sim Q) \vee (Q \wedge P) \vee \cancel{(Q \wedge \sim Q)}$$

$$\equiv (\sim P \wedge \sim Q) \vee (Q \wedge P)$$

<u>Proving a biconditional:</u>

Typically: to prove $P \iff Q$, prove $P \Rightarrow Q$

then prove $Q \Rightarrow P$

<u>Exercise:</u> Prove for every integer n,

$$n \text{ is odd} \iff n^2 \text{ is odd}$$

Hint: one direction is easier.

Proof: $(\Rightarrow)$ Suppose n is odd. Then $n = 2a+1$ for some $a \in \mathbb{Z}$

$$n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 2(2a^2+2a)+1$$

which is odd b/c $2a^2+2a \in \mathbb{Z}$.

$(\Leftarrow)$ We want to show n^2 is odd $\Rightarrow$ n is odd.

Contrapositive: n^2 is even $\Leftarrow$ n is even

We prove the contrapositive.

If n is even then $n = 2a$, $a \in \mathbb{Z}$

$$n^2 = 4a^2 = 2\cdot(2a^2) \text{ is even. } \blacksquare$$

Contrapositive (Ch 5.1): Recall $\sim Q \Rightarrow \sim P$ is the contrapositive of $P \Rightarrow Q$. It is logically equivalent.

(n is odd $\iff n^2$ is odd)

Another possible proof: $P \iff Q \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$

Case 1: n is odd. Then $n = 2a+1$

$\rightsquigarrow n^2 = 4a^2 + 4a + 1 = 2(2a^2+2a)+1$ is odd.

So $\underbrace{n \text{ is odd}}_{T}$ and $\underbrace{n^2 \text{ is odd}}_{T}$

Case 2: n is even. Then $n = 2a$

$\rightsquigarrow n^2 = (2a)^2 = 2(2a^2)$ is even

So $\underbrace{n \text{ is odd}}_{F}$. and $\underbrace{n^2 \text{ is odd}}_{F}$.

Exercise: Prove for every $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$

ab is even $\iff$ a is even or b is even.

Exercise: For any positive real numbers x and y,

$$x > y \iff x^2 > y^2$$

(You can assume if $c > 0$, then $(a > b) \iff (ac > bc)$)

Proposition: For every pair of integers a, b

$$ab \text{ is even} \iff a \text{ is even or } b \text{ is even.}$$

Pf: $(\Rightarrow)$ We prove the contrapositive:

"a is odd and b is odd $\Rightarrow$ ab is odd"

Suppose a is odd and b is odd. Write $a = 2c+1$ for some $c \in \mathbb{Z}$, $b = 2d+1$ for some $d \in \mathbb{Z}$.

Then $ab = (2c+1)(2d+1) = 4cd + 2c + 2d + 1$
$= 2(2cd + c + d) + 1$ which is odd.

$(\Leftarrow)$ Suppose either a is even or b is even.

Case 1: a is even. Then $a = 2c$ for some $c \in \mathbb{Z}$

$$ab = 2c \cdot b = 2 \cdot (bc) \quad \text{is even.}$$

Case 2: b is even. This is similar to Case 1. ∎

Proposition: For any positive real numbers x and y, $x > y \iff x^2 > y^2$

Pf: $(\Rightarrow)$. Suppose $x > y$. Then $x \cdot x > x \cdot y$ and $y \cdot x > y \cdot y$

So $x^2 > xy > y^2$. Therefore, $x^2 > y^2$

$(\Leftarrow)$ We prove the contrapositive. Suppose $x \le y$. Either $x < y$ or $x = y$.

Case 1: $x < y$. Then $y > x$, and then by the same argument as was used to prove $(\Rightarrow)$, $y^2 > x^2$

Case 2: $x = y$. Then $x \cdot x = xy$ and $y \cdot x = y \cdot y$

In both cases, $x^2 \le y^2$

So $x^2 = xy = y^2$. Therefore $x^2 = y^2$. ∎